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ASSESSING POSITION ORDER IN RIGID BODY GUIDANCE: AN INTUITIVE APPROACH TO FIXED PIVOT SELECTION

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ABSTRACT

This paper presents a new method for determining whether an RR dyad will pass through a set of finitely separated positions in order. Several established solution methods have been previously documented for this problem. This method utilizes only the displacement poles in the fixed frame to assess in an intuitive fashion whether a selected fixed pivot location will result in an ordered dyad solution. A line passing through the selected fixed pivot is rotated one-half revolution about the fixed pivot, in a manner similar to a propeller with infinitely long blades, to sweep the entire plane. Order is established by tracking the sequence of the displacement poles intersected by the rotating line. With four or five positions, fixed pivot locations corresponding to dyads having any specified order are readily found. Five-position problems can be directly evaluated to determine if any ordered solutions exist, and degenerate cases of four positions for which the set of fixed pivots corresponding to ordered dyads collapses to a single point on the center point curve can be identified.

INTRODUCTION

As detailed by Chase and Mirth [2], linkage synthesis techniques can produce unacceptable solutions that suffer from any of three defects. With a circuit defect, the linkage must be disconnected and reassembled to achieve all positions. This defective solution is associated with linkage configurations that lie on two distinct coupler curves. With a branch defect, no single crank can be actuated to move the linkage through all positions. This defective solution occurs when the crank orients the linkage into a stationary, or toggle, orientation between the given positions. The order defect, which is the focus of this paper, occurs when the coupler moves through the design positions in the wrong order, as the crank is driven in a uniform direction. While the branch and circuit defects are associated with an entire mechanism, the order defect is related to the driving dyad.

Solution rectification refers to the elimination of defects from a mechanism synthesis result. Filemon [4] pioneered linkage solution rectification, outlining graphical construction techniques for the four position synthesis of a crank-rocker, which eliminate order and circuit defects. Waldron [9, 10] further addressed the order problem by mapping permissible segments of the circle-point curve that, when synthesized, obtain a dyad that achieves the design positions in order. Prentis [8] presented a process to delineate areas of a four position problem within which the crank center must lie based on how the potential crank center views the displacement poles. This paper presents a new look at whether a single dyad will achieve design positions in order.

The remainder of this paper is organized as follows. Crank angles are defined and necessary conditions for an ordered dyad are presented. The “Center Point Theorem” of McCarthy [5] is used to establish necessary and sufficient conditions for the relative location of the poles to a fixed pivot for an ordered dyad. Using these concepts, a “propeller method” for determining the order that is associated with a fixed pivot location is presented. Lastly, several illustrative examples are given.

ORDERED DYADS

During discussion of this approach, a counter-clockwise crank rotation is assumed. With counter-clockwise rotation assumed to be positive, all crank angles are measured such that

$$0 < \beta_{ij} < 2\pi \quad \forall i, j \tag{1}$$

where β_{ij} is the change in crank angle from position i to j. For an RR chain to achieve target positions in a specific order the driving crank must move in the correct sequence without reversing directions. That is, the crank angle β measured counterclockwise from the first position, must monotonically increase as each target position is reached. Therefore, a dyad does not exhibit an order defect if

$$0 < \beta_{12} < \beta_{13} < \beta_{14} < \ldots < \beta_{1N} < 2\pi. \tag{2}$$

CENTER-POINT THEOREM

The center-point theorem, as developed in this section, specifies a relationship between crank angles with a fixed pivot and points generated from the target positions. This relationship is a fundamental concept for synthesizing an RR dyad that achieves target positions in order

Any displacement of a rigid body from position i to position j can be accomplished by a pure rotation about the displacement pole, $\mathbf{P_{ij}}$. Clearly, this point is also the center of pure rotation of the displacement from position j to position i. Therefore, $\mathbf{P_{ij}} = \mathbf{P_{ji}}$.

Figure 1 illustrates the floating link of an RR chain displaced from position M_i to position M_j under a crank rotation of β_{ij}. Note the pair of triangles formed by the pole, $\mathbf{P_{ij}}$, the fixed pivot, **G**, and the moving pivot, **W**. McCarthy [5] termed these dyad triangles. By definition, the pole $\mathbf{P_{ij}}$ lies on the perpendicular bisector of $\mathbf{W_i}$ and $\mathbf{W_j}$. Therefore, the interior angle of each dyad triangle at **G** is half of the crank angle β_{ij}. This is shown in Figure 1 as

$$\angle \mathbf{W_i G P_{ij}} = \angle \mathbf{P_{ij} G W_j} = \frac{\beta_{ij}}{2}. \tag{3}$$

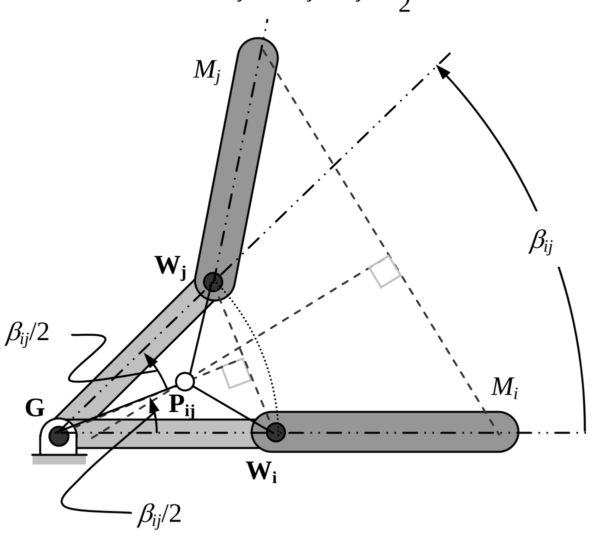


Figure 1: The relationship between the fixed pivot of two positions, the pole and the crank angle

The displacement of the floating link also could be such that the pole is located on the opposite side of the fixed pivot. That situation is depicted in Figure 2 wherein

$$\angle \mathbf{W_i G P_{ij}} = \angle \mathbf{P_{ij} G W_j} = \frac{\beta_{ij}}{2} + \pi. \tag{4}$$

Summarizing the two cases in Equations 3 and 4:

$$\angle \mathbf{W_i G P_{ij}} = \angle \mathbf{P_{ij} G W_j} = \frac{\beta_{ij}}{2} \text{ or } \frac{\beta_{ij}}{2} + \pi \tag{5}$$

Figure 3 examines the geometry as a third position M_k is included. The angle between the precision points and the pole as viewed from the fixed pivot are combined.

$$\angle \mathbf{P_{ij} G P_{jk}} = \angle \mathbf{P_{ij} G W_j} + \angle \mathbf{W_j G P_{jk}} \tag{6}$$

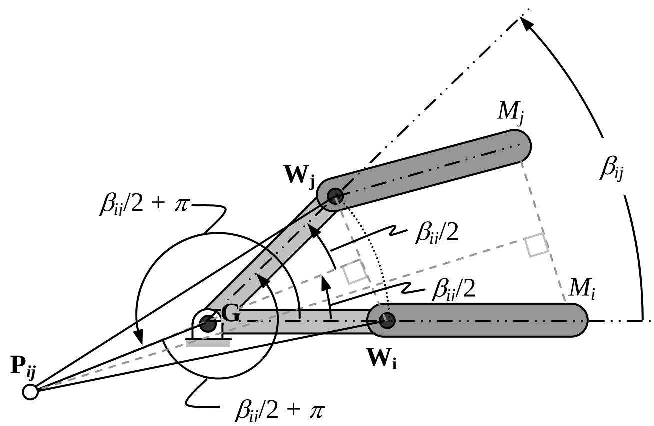


Figure 2: A second possible relationship between the fixed pivot of two positions, the pole and the crank angle.

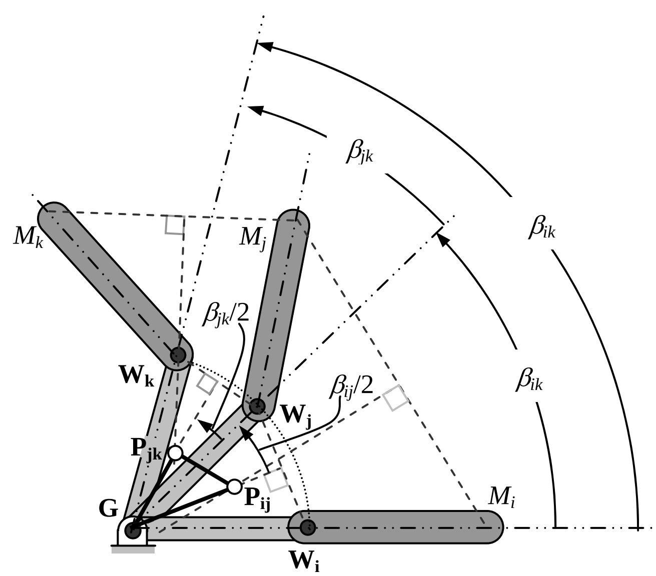


Figure 3: The relationship between the fixed pivot, the poles and three positions.

Applying Equation 5 to Equation 6 yields

$$\angle \mathbf{P_{ij}GP_{jk}} = \frac{\beta_{ij}}{2} + \frac{\beta_{jk}}{2} \text{ or } \frac{\beta_{ij}}{2} + \frac{\beta_{jk}}{2} + \pi \,. \tag{7}$$

Using the addition of crank angles $\beta_{ij} + \beta_{jk} = \beta_{ik}$ results in the center point theorem, as stated by McCarthy [2]:

$$\angle \mathbf{P_{ij}GP_{jk}} = \frac{\beta_{ik}}{2} \text{ or } \frac{\beta_{ik}}{2} + \pi \tag{8}$$

recalling that either value is positive counter-clockwise.

Equation 8 expresses a relationship between the crank angle, the poles, and the fixed pivot location. In the following sections, Equations 2 and 8 will be combined to generate a method for determining the position order of a synthesized dyad.

A NECESSARY CONDITION ON THE POLES

Figure 4 illustrates two successive poles, and a fixed pivot. Note that angle of rotation about **G** from the line segment $\overline{\mathbf{GP_{ij}}}$ to the line segment $\overline{\mathbf{GP_{jk}}}$ is $\frac{\beta_{ik}}{2} + \pi \,.$

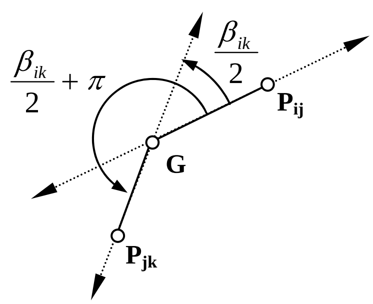


Figure 4: The angle between $\overleftrightarrow{\mathbf{GP_{ij}}}$ and $\overleftrightarrow{\mathbf{GP_{jk}}}$.

Let $\overleftrightarrow{\mathbf{GP_{ij}}}$ be a line through points **G** and $\mathbf{P_{ij}}$, extending infinitely in both directions. Note that angle of rotation about **G** from the line $\overleftrightarrow{\mathbf{GP_{ij}}}$ to the line $\overleftrightarrow{\mathbf{GP_{jk}}}$ is $\frac{\beta_{ik}}{2}$, and

$$\angle \overleftrightarrow{\mathbf{GP_{ij}}}\overleftrightarrow{\mathbf{GP_{jk}}} = \frac{\beta_{ik}}{2} < \pi \tag{9}$$

From Equation 2, a dyad exhibits no order problem if

$$0 < \frac{\beta_{12}}{2} < \frac{\beta_{13}}{2} < \ldots < \pi \,. \tag{10}$$

Combining Equations 9 and 10, a dyad exhibits no order problem if

$$0 < \angle \overleftrightarrow{\mathbf{GP_{1k}}}\overleftrightarrow{\mathbf{GP_{2k}}} < \angle \overleftrightarrow{\mathbf{GP_{1k}}}\overleftrightarrow{\mathbf{GP_{3k}}} < \ldots < \angle \overleftrightarrow{\mathbf{GP_{1k}}}\overleftrightarrow{\mathbf{GP_{Nk}}} < \pi \tag{11}$$

Interpreted graphically, line $\overleftrightarrow{\mathbf{GP_{1k}}}$ rotates counter-clockwise about **G** analogous to a propeller with infinitely long blades rotating about a fixed pivot. A necessary condition for an ordered dyad with a fixed pivot at G then is that the line begins at $\mathbf{P_{1k}}$ and sweeps through the poles $\mathbf{P_{2k}}$, $\mathbf{P_{3k}}$, …, $\mathbf{P_{Nk}}$ in order within a rotation of π radians.

A SUFFICIENT CONDITION ON THE POLES

The above condition is necessary, but not sufficient. Consider the poles relative to position *k* in Equation 11.

$$\ldots < \angle \overleftrightarrow{\mathbf{GP_{1k}}}\overleftrightarrow{\mathbf{GP_{k-1\,k}}} < \angle \overleftrightarrow{\mathbf{GP_{1k}}}\overleftrightarrow{\mathbf{GP_{k\,k}}} < \angle \overleftrightarrow{\mathbf{GP_{1k}}}\overleftrightarrow{\mathbf{GP_{k\,k+1}}} < \ldots < \pi \,. \tag{12}$$

, which includes the undefined quantity $\mathbf{P_{kk}}$. Therefore, Equation 11 contains no information that confirms the location of position *k* within the order of the other positions. Observe, however, that Equation 11 will be true for all *k*. Hence, a sufficient condition for an ordered dyad is

$$k=1: \quad 0 < \angle \overleftrightarrow{\mathbf{GP_{12}}}\overleftrightarrow{\mathbf{GP_{13}}} < \angle \overleftrightarrow{\mathbf{GP_{12}}}\overleftrightarrow{\mathbf{GP_{14}}} < \ldots < \angle \overleftrightarrow{\mathbf{GP_{12}}}\overleftrightarrow{\mathbf{GP_{1N}}} < \pi \,, \tag{13}$$

$$k=2: \quad 0 < \angle \overleftrightarrow{\mathbf{GP_{12}}}\overleftrightarrow{\mathbf{GP_{23}}} < \angle \overleftrightarrow{\mathbf{GP_{12}}}\overleftrightarrow{\mathbf{GP_{24}}} < \ldots < \angle \overleftrightarrow{\mathbf{GP_{12}}}\overleftrightarrow{\mathbf{GP_{2N}}} < \pi \,, \tag{14}$$

$$k=3: \quad 0 < \angle \overleftrightarrow{\mathbf{GP_{13}}}\overleftrightarrow{\mathbf{GP_{23}}} < \angle \overleftrightarrow{\mathbf{GP_{13}}}\overleftrightarrow{\mathbf{GP_{34}}} < \ldots < \angle \overleftrightarrow{\mathbf{GP_{13}}}\overleftrightarrow{\mathbf{GP_{3N}}} < \pi \,, \tag{15}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$k=N: \quad 0 < \angle \overleftrightarrow{\mathbf{GP_{1N}}}\overleftrightarrow{\mathbf{GP_{2N}}} < \angle \overleftrightarrow{\mathbf{GP_{1N}}}\overleftrightarrow{\mathbf{GP_{3N}}} < \ldots < \angle \overleftrightarrow{\mathbf{GP_{1N}}}\overleftrightarrow{\mathbf{GP_{N-1\,N}}} < \pi \,. \tag{16}$$

The following may be observed. As a line, infinitely extending in both directions rotates about **G**, starting at $\mathbf{P_{1k}}$, it sweeps through every pole in the plane in a counter-clockwise turn of π radians. If every other pole involving position *k* is encountered in ascending order, and this is true for all *k*, then the positions are ordered for that choice of fixed pivot **G**. This rotating line is analogous to a propeller, with infinitely long blades, rotating about **G** and sweeping through the poles. Inspecting the poles to assess order in this fashion is termed the propeller method.

For an ordered solution to a three position synthesis problem with a counter-clockwise crank, the line starts at $\overleftrightarrow{\mathbf{GP_{12}}}$ and must intersect, in order, $\mathbf{P_{13}}$ and $\mathbf{P_{23}}$. This is represented with the notation $\mathbf{P_{12}}$ $\mathbf{P_{13}}$ $\mathbf{P_{23}}$, meaning that if the line intersects the poles in that order, as it rotates over a counter-clockwise turn or π, the crank is ordered.

For an ordered solution to a four position synthesis problem with a counter-clockwise crank, the line starting at $\overleftrightarrow{\mathbf{GP_{12}}}$ and rotating counter-clockwise must intersect $\mathbf{P_{13}}$ then $\mathbf{P_{14}}$ ($\mathbf{P_{12}}$ $\mathbf{P_{13}}$ $\mathbf{P_{14}}$). When starting at $\overleftrightarrow{\mathbf{GP_{12}}}$ the line must next intersect $\mathbf{P_{23}}$ then $\mathbf{P_{24}}$ ($\mathbf{P_{12}}$ $\mathbf{P_{23}}$ $\mathbf{P_{24}}$). When starting at $\overleftrightarrow{\mathbf{GP_{13}}}$ the line must also intersect $\mathbf{P_{23}}$ then $\mathbf{P_{34}}$ or $\mathbf{P_{13}}$ $\mathbf{P_{23}}$ $\mathbf{P_{34}}$. Finally, when starting at $\overleftrightarrow{\mathbf{GP_{14}}}$ it must also intersect $\mathbf{P_{24}}$ then $\mathbf{P_{34}}$ ($\mathbf{P_{14}}$ $\mathbf{P_{24}}$ $\mathbf{P_{34}}$). These statements can be combined to yield $\mathbf{P_{12}}$ $\mathbf{P_{13}}$ $\mathbf{P_{14}}$ $\mathbf{P_{23}}$ $\mathbf{P_{24}}$ $\mathbf{P_{34}}$, $\mathbf{P_{12}}$ $\mathbf{P_{13}}$ $\mathbf{P_{23}}$ $\mathbf{P_{14}}$ $\mathbf{P_{24}}$ $\mathbf{P_{34}}$, $\mathbf{P_{12}}$ $\mathbf{P_{34}}$ $\mathbf{P_{13}}$ $\mathbf{P_{14}}$ $\mathbf{P_{23}}$ $\mathbf{P_{24}}$, $\mathbf{P_{12}}$ $\mathbf{P_{34}}$ $\mathbf{P_{13}}$ $\mathbf{P_{23}}$ $\mathbf{P_{14}}$ $\mathbf{P_{24}}$, as acceptable orders for a four position synthesis problem in a single propeller turn of π.

For five positions, an ordered dyad will sweep through the poles in the following orders: $\mathbf{P_{12}}$ $\mathbf{P_{13}}$ $\mathbf{P_{14}}$ $\mathbf{P_{15}}$, $\mathbf{P_{12}}$ $\mathbf{P_{23}}$ $\mathbf{P_{24}}$ $\mathbf{P_{25}}$, $\mathbf{P_{13}}$ $\mathbf{P_{23}}$ $\mathbf{P_{34}}$ $\mathbf{P_{35}}$, $\mathbf{P_{14}}$ $\mathbf{P_{24}}$ $\mathbf{P_{34}}$ $\mathbf{P_{45}}$, and $\mathbf{P_{15}}$ $\mathbf{P_{25}}$ $\mathbf{P_{35}}$ $\mathbf{P_{45}}$. Combining the propeller results generates 54 possibilities for an ordered dyad with a five position problem.

THE PROPELLER METHOD

From Equation 13, the crank angle relationship

$$0 < \beta_{23} < \beta_{24} < \ldots < \beta_{2N} < 2\pi \tag{17}$$

is established. For k=1 alone, positions 3 through N can be confirmed to be ordered for the crank. Also, position 2 lies, as desired, somewhere between position N and 3. All that is required to confirm an ordered solution is to determine that position 1 also occurs between N and 3 and before 2. This is verified by establishing

$$0 < \beta_{12} < \beta_{13} < \beta_{1N} < 2\pi. \tag{18}$$

Note that if the rotating line intersects $\mathbf{P_{12}}$ $\mathbf{P_{13}}$ $\mathbf{P_{23}}$ in that order, $\beta_{23} < \beta_{13}$, establishing that $\beta_{12} < \beta_{13}$. Moreover, if the rotating line intersects $\mathbf{P_{12}}$ $\mathbf{P_{23}}$ $\mathbf{P_{2N}}$ in that order, $\beta_{13} < \beta_{1N}$. Thus, Equations 17 and 18, expressed as relationships among poles via McCarthy's Center Point Theorem, offer proof of the following.

Theorem: As a line, infinitely extending in both directions, rotates about **G**, it sweeps through every pole in the plane in a counter-clockwise (clockwise) turn of π radians. Starting when the line passes through $\mathbf{P_{12}}$, if it intersects $\mathbf{P_{13}}$ $\mathbf{P_{14}}$ **…** $\mathbf{P_{1N}}$ in order, and intersects $\mathbf{P_{13}}$ $\mathbf{P_{23}}$ $\mathbf{P_{2N}}$ in order, then the positions are ordered for that choice of fixed pivot **G** with a counter-clockwise (clockwise) rotation

The entire development of this theorem assumed a counter-clockwise crank rotation. The derivation may be readily performed by considering the crank rotating clockwise, arriving at the clockwise version of the theorem.

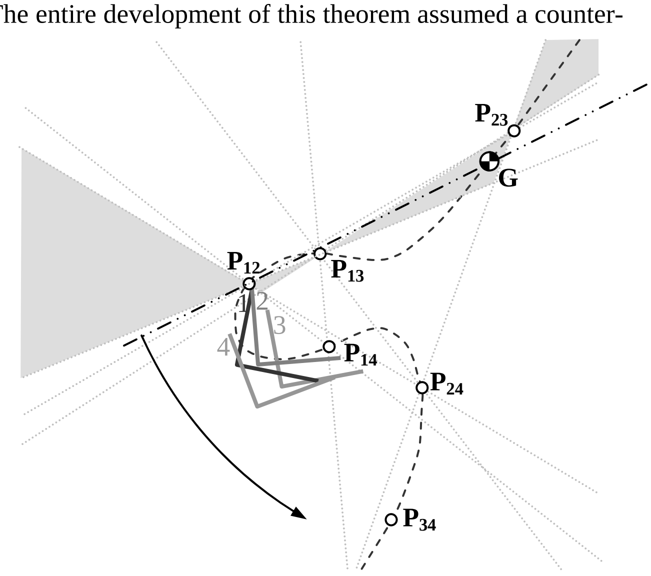
For four positions, the order check reduces to $\mathbf{P_{12}}$ $\mathbf{P_{13}}$ $\mathbf{P_{14}}$ and $\mathbf{P_{12}}$ $\mathbf{P_{13}}$ $\mathbf{P_{23}}$ $\mathbf{P_{24}}$. For five positions, the order check reduces to $\mathbf{P_{12}}$ $\mathbf{P_{13}}$ $\mathbf{P_{14}}$ $\mathbf{P_{15}}$ and $\mathbf{P_{12}}$ $\mathbf{P_{13}}$ $\mathbf{P_{23}}$ $\mathbf{P_{25}}$.

EXAMPLES

Four Precision Points:

Figure 5a illustrates four precision points, the associated displacement poles, the center point curve, and shaded regions that satisfy the propeller method. In addition, a fixed pivot location is selected in a shaded region. A line, extending infinitely in both directions, is placed at the fixed pivot, and directed through $\mathbf{P_{12}}$ is shown. As the line is rotated counter-clockwise, it sweeps through $\mathbf{P_{13}}$, then $\mathbf{P_{23}}$ with the other end, and $\mathbf{P_{14}}$ $\mathbf{P_{24}}$ $\mathbf{P_{34}}$. The propeller method declares that with this ordering of the poles, the crank of a synthesized dyad will achieve the four positions in proper order. The resulting dyad is shown in Figure 5b. Clearly, as the crank rotates monotonically in a counterclockwise direction the precision points are achieved in order.

Figure 6a illustrates a second case of four precision points, the poles and the center point curve. In this example, two poles, $\mathbf{P_{13}}$ and $\mathbf{P_{24}}$, are coincident. For this case, the propeller method makes apparent the problem associated with $\mathbf{P_{13}}$ and $\mathbf{P_{24}}$ being coincident. Whenever the rotating line strikes $\mathbf{P_{13}}$, it also intersects $\mathbf{P_{24}}$. The propeller method dictates that $\mathbf{P_{23}}$ must fall between $\mathbf{P_{13}}$ and $\mathbf{P_{24}}$. Therefore, the only valid fixed pivots must be placed along the line defined by $\mathbf{P_{13}}$, $\mathbf{P_{24}}$, and $\mathbf{P_{23}}$ That



(a) Positions, poles, center point curve, ordered regions, fixed pivot and propeller

(b) Ordered dyad solution

Figure 5: A four position application of the propeller method

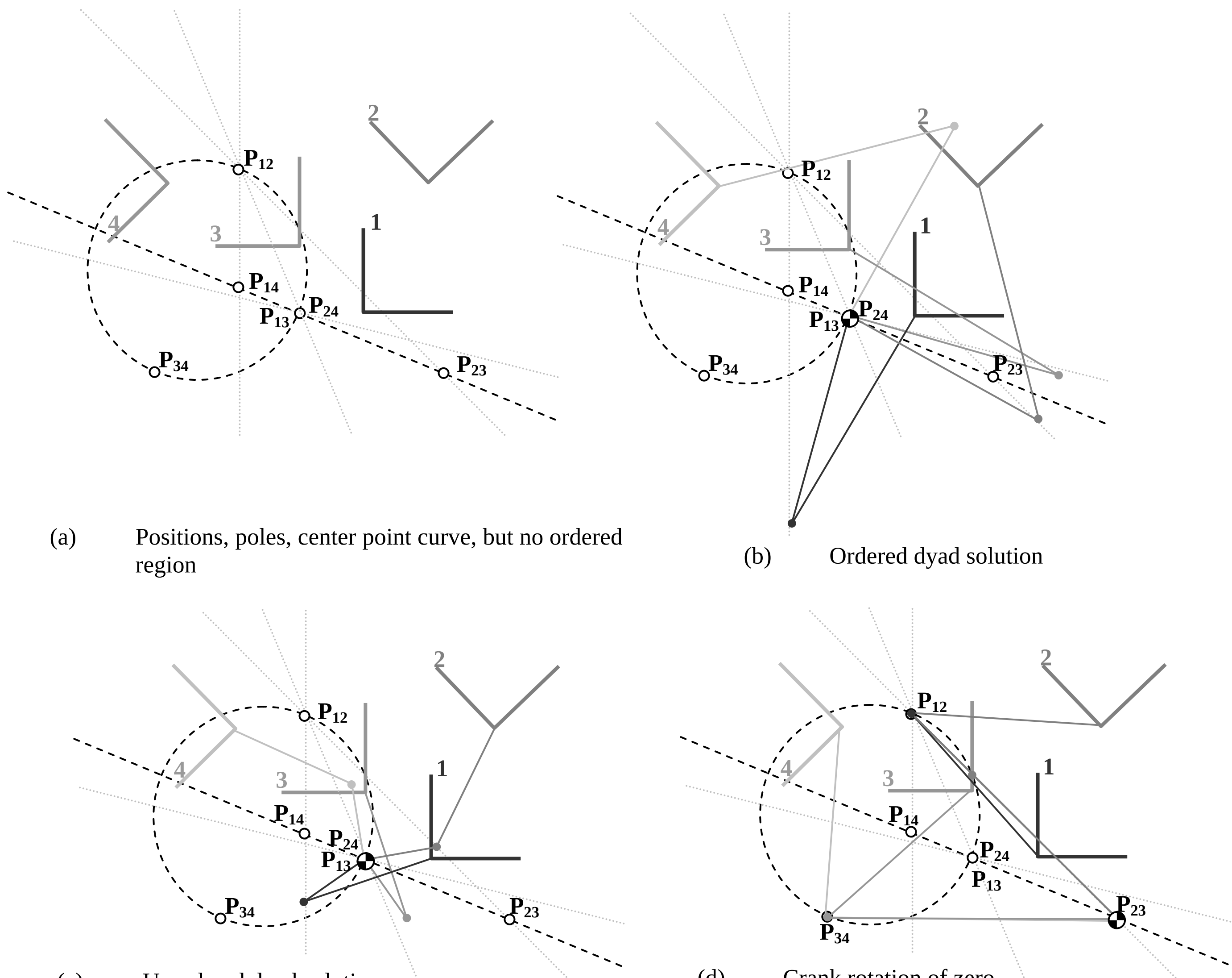
way, all three poles are swept at the same time by the rotating line.

When $\mathbf{P_{13}}$ is selected as a fixed pivot, multiple dyad solutions are possible. A dyad that generates an ordered solution is shown in figure 6b. Figure 6c shows another dyad from the same fixed pivot that produces an unordered solution. Finally, figure 6d shows a dyad solution with $\mathbf{P_{23}}$ selected as the fixed pivot. Notice that crank rotation of zero is generated.

Reviewing this example, the propeller method provided a scheme for readily identifying situations with very restricted possibilities for an ordered dyad.

Five Precision Points:

Figure 7a illustrates five precision points, the displacement poles and shaded regions that satisfy the propeller method. A feasible fixed pivot location is shown, consistent with five position synthesis techniques. Also, a line is shown extending infinitely in both directions, placed at the fixed pivot and directed through $\mathbf{P_{12}}$. As the line is rotated counter-clockwise, it sweeps through $\mathbf{P_{35}}$ $\mathbf{P_{45}}$ $\mathbf{P_{13}}$ $\mathbf{P_{14}}$ $\mathbf{P_{23}}$ $\mathbf{P_{24}}$ $\mathbf{P_{15}}$ $\mathbf{P_{34}}$ $\mathbf{P_{25}}$. The propeller method declares that the crank of a dyad will achieve the five positions in proper order. The resulting dyad is shown in figure 7b.



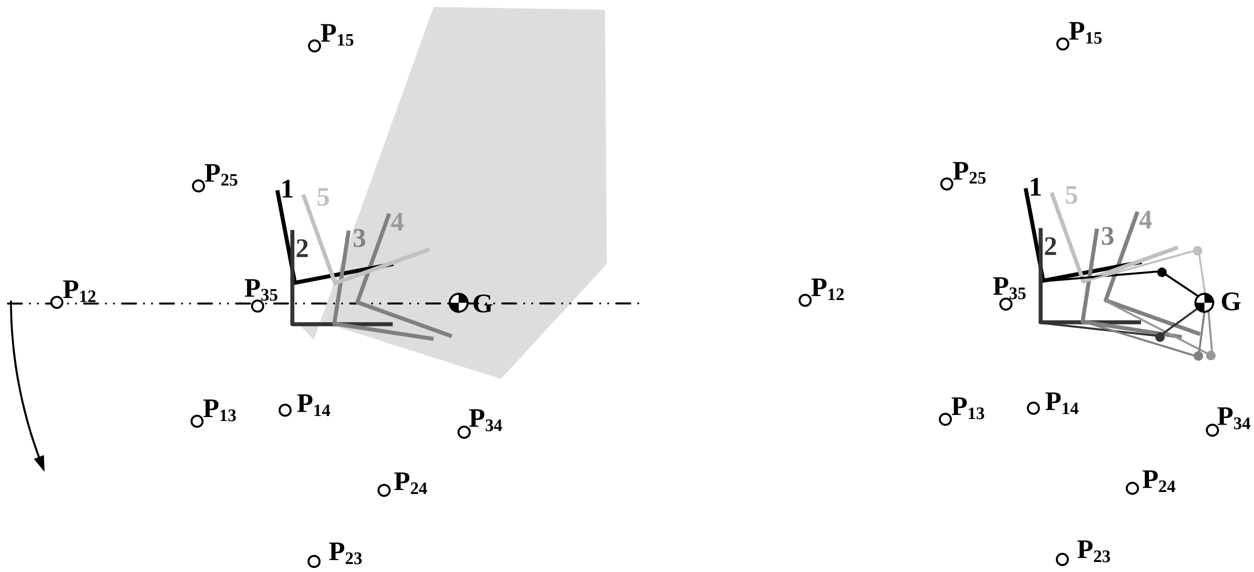
(a) Positions, poles, center point curve, but no ordered region

(b) Ordered dyad solution

(c) Unordered dyad solution

(d) Crank rotation of zero

Figure 6: A special four position application of the propeller method



(a) Positions, poles, ordered region, fixed pivot and propeller

(b) Ordered dyad solution

Figure 7: A five position application of the propeller method

Several five position point cases have been detected where no fixed pivots are able to satisfy the propeller method. Thus, no ordered solutions exist on the entire plane. Figure 8 illustrates such a case.

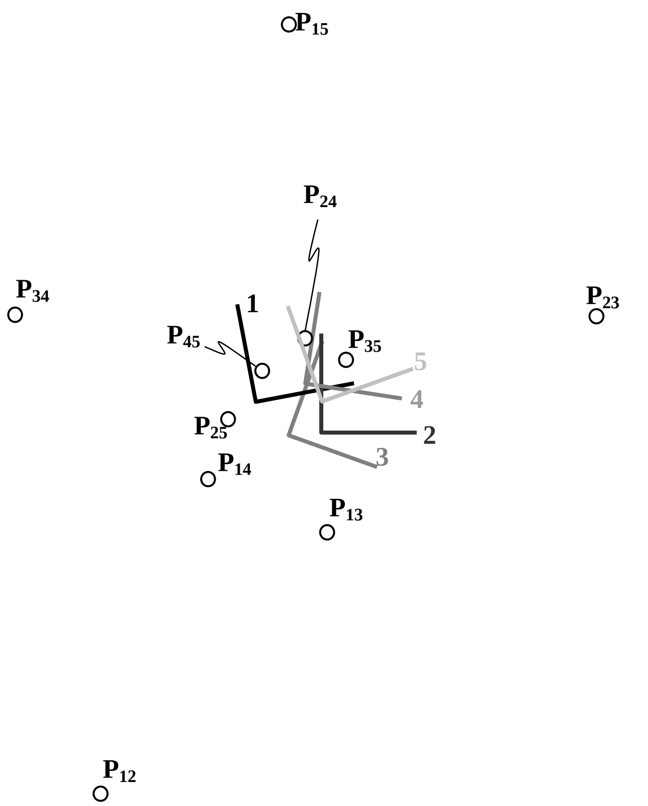


Figure 8: Five position without any ordered regions

CONCLUSION

The order by which any number of positions is reached with an RR dyad depends solely on the relative location of the fixed pivot and the displacement poles. A simple check on whether finitely separated positions will be reached in order was presented. This check is based on a propeller approach, which was substantiated from fundamental theories of displacement poles. Additionally, necessary and sufficient conditions for an ordered solution were developed, simplifying the order check.

The propeller method was developed based on a counter-clockwise rotation of a line that passes through the fixed pivot. The line, or propeller, must encounter the displacement poles in a specific order to assure that the positions will be reached, in order, with a counter-clockwise rotation of the crank. Conversely, the propeller method can be used with a clockwise rotation, verifying an ordered solution for a clockwise crank rotation.

ACKNOWLEDGMENTS

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